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Prescriptionless light-cone integrals

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Abstract. Perturbative quantum gauge field theory as seen within the perspective of physical gauge choices such as the light-cone gauge entails the emergence of troublesome poles of the type $(k \cdot n)^{-\alpha}$ in the Feynman integrals. These come from the boson field propagator, where $\alpha = 1, 2, \cdots$ and n^μ is the external arbitrary four-vector that defines the gauge proper. This becomes an additional hurdle in the computation of Feynman diagrams, since any graph containing internal boson lines will inevitably produce integrands with denominators bearing the characteristic gauge-fixing factor. How one deals with them has been the subject of research over decades, and several prescriptions have been suggested and tried in the course of time, with failures and successes. However, a more recent development at this fronteer which applies the negative dimensional technique to compute light-cone Feynman integrals shows that we can altogether dispense with prescriptions to perform the calculations. An additional bonus comes to us attached to this new technique, in that not only it renders the light-cone prescriptionless but, by the very nature of it, it can also dispense with decomposition formulas or partial fractioning tricks used in the standard approach to separate pole products of the type $(k \cdot n)^{-\alpha}[(k-p) \cdot n]^{-\beta}$ $(\beta = 1, 2, \cdots)$. In this work we demonstrate how all this can be done.

1 Introduction

The light-cone gauge for gauge field theories is probably one of the most widely used among the algebraic non-covariant gauges. Its popularity has known ups and downs along its history. Among the ups are that the emerging propagator has a deceivingly simple structure compared to other non-covariant choices, the decoupling of Faddeev–Popov ghosts from the physical fields, and the possibility of describing and modeling complex supersymmetric string theories in it. The ugly side of the coin is represented by the subtle $(k \cdot n)^{-\alpha}$ singularities present in all the physical amplitudes described within it. Such a complication demanded ad hoc prescriptions to handle the singularity in a mathematically consistent way. Apart from the fact that such an expedient has to applied by hand, it was soon realized that it was not enough to be mathematically well-defined; it had to be physically consistent as well. Thus, not any prescription is suitable, but only *causal* prescriptions are eligible for the light-cone gauge.

Probably the major breakthrough in recent years along this line is the realization that D-dimensional Feynman integrals can be analytically continued to negative dimensions to be performed there and then can be brought back to a positive dimensionality [1,2]. Negative dimensional integration method (NDIM) is tantamount to the performing of fermionic integration in positive dimensions [3]. This can be applied to light-cone integrals with surprising effects. No prescription is called for in the computation [4] and moreover, as shortly can be seen, it can dispense altogether with the necessity of partial fractioning products of gauge-dependent poles [5], a condition sine qua non when one resorts to the use of prescriptions.

In this work we shall demonstrate the two surprising features of NDIM when employed in the light-cone context: no prescriptions and no partial fractionings are needed. Our lab-testing is performed taking the simplest scalar and tensorial structures for one-loop integrals.

2 One-loop light-cone gauge integrals

First of all, let us make things more concrete, by analyzing the framework of vector gauge fields, e.g. the pure Yang–Mills fields, where, after taking the limit of a vanishing gauge parameter, the propagator reads

$$D^{ab}_{\mu\nu}(k) = \frac{-\mathrm{i}\delta^{ab}}{k^2 + \mathrm{i}\varepsilon}\left[g_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{k \cdot n}\right], \qquad (1)$$

where (a,b) are the gauge group indices, n_μ is the arbitrary and constant light-like four-vector which defines the gauge, $n \cdot A^a(x) = 0$; $n^2 = 0$. This propagator generates D-dimensional Feynman integrals of the following generic form:

$$I_{lc} = \int \frac{\mathrm{d}^D k_i}{A(k_j, p_l)} \frac{f(k_j \cdot n^*, p_l \cdot n^*)}{h(k_j \cdot n, p_l \cdot n)}, \qquad (2)$$

where p_l labels all the external momenta, and n^*_μ is a null four-vector, dual to n_μ. A conspicuous feature that

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we need to note first of all is that the dual vector n^*_μ, when it appears at all, it does so *always* and *only* in the numerators of the integrands. And herein comes the first seemingly "mysterious" facet of the light-cone gauge. Why is it that from a propagator expression like (1), which contains no n^* factors there can arise integrals of the form (2), with n^* factors prominently seen? Again, this is most easily seen in the framework of definite external vectors n and n^*. An alternative way of writing the generic form of a light-cone integral is

$$I_{lc}^{\mu_1\cdots\mu_n} = \int \frac{\mathrm{d}^D k_i}{A(k_j,p_l)} \frac{g(k_j^{\mu_j},p_l^{\mu_l})}{h(k_j\cdot n, p_l\cdot n)}, \tag{3}$$

where the numerator $g(k_j^{\mu_j},p_l^{\mu_l})$ defines the tensorial structure in the integral. For a vector, we have $k^\mu = (k^+,k^-,\mathbf{k}^t)$, where $k^+ = 2^{-1/2}(k^0+k^{D-1})$ and $k^- = 2^{-1/2}(k^0-k^{D-1})$. If we choose definite n and n^* such that $n_\mu = (1,0,\cdots,1)$, and $n^*_\mu = (1,0,\cdots,-1)$, this allows us to write $k^+ \equiv k\cdot n$ and $k^- \equiv k\cdot n^*$. We have therefore traced the origin for the numerator factors containing n^*. We would like to emphasize here that the presence of this n^* in the numerators of integrands has nothing whatsoever to do with some kind of prescription input. It is rather an intrinsic feature of the general structure of a Feynman integral in the light-cone gauge.

Of course, for practical reasons we illustrate the NDIM methodology picking up only a few of the scalar, vector and second-rank tensor one-loop integrals. So, we shall be considering the following:

$$T_1(i,j,l) = \int \mathrm{d}^D q \mathbf{N}(q), \tag{4}$$

$$T_1^\mu(i,j,l) = \int \mathrm{d}^D q q^\mu \mathbf{N}(q), \tag{5}$$

$$T_1^{\mu\nu}(i,j,l) = \int \mathrm{d}^D q q^\mu q^\nu \mathbf{N}(q), \tag{6}$$

where

$$\mathbf{N}(q) \equiv [(q-p)^{2i}](q\cdot n)^j (q\cdot n^*)^l$$

and

$$T_2(i,j,l,m) = \int \mathrm{d}^D q \mathbf{R}(q), \tag{7}$$

$$T_2^\mu(i,j,l,m) = \int \mathrm{d}^D q q^\mu \mathbf{R}(q), \tag{8}$$

$$T_2^{\mu\nu}(i,j,l,m) = \int \mathrm{d}^D q q^\mu q^\nu \mathbf{R}(q), \tag{9}$$

where

$$\mathbf{R}(q) \equiv [(q-p)^{2i}](q\cdot n)^j [(q-p)\cdot n]^l (q\cdot n^*)^m .$$

In the first three type T_1 integrals, after they are computed in NDIM, only the exponents (i,j) will be analytically continued to allow for negative values, since the original structure of the Feynman integral demands an exponent $l \geq 0$. Similarly, for the last three type T_2 integrals only the exponents (i,j,l) will be analytically continued to negative values, whereas $m \geq 0$. We strongly emphasize this point in view of the fact that we must respect the very nature of the original structure for the light-cone integrals, where factors of the form $(q\cdot n^*)$ *never* appear in the denominators.

Observe that we are not invoking any kind of prescription for the $(q\cdot n)^j$ factors to solve the integrals in NDIM, since before analytic continuation j is strictly *positive* and there are no poles to circumvent! This is the beauty and strength of NDIM! Neither are the $(q\cdot n^*)^l$ numerator factors due to some sort of prescription input as they are, e.g., in the Mandelstam–Leibbrandt (ML) treatment, where one makes the substitution [6–11]

$$\mathcal{M} = \int \frac{\mathrm{d}^D q}{(q-p)^2 (q\cdot n)} \xrightarrow{\mathrm{ML}} \int \frac{\mathrm{d}^D q (q\cdot n^*)}{(q-p)^2 \left[(q\cdot n)(q\cdot n^*)+\mathrm{i}\epsilon\right]}. \tag{10}$$

Let us then evaluate the integrals using the NDIM approach. In fact, our first integral T_1 has already been calculated in great detail in our previous paper [4] the result of which is

$$T_1^{\mathrm{AC}}(i,j,l) = \pi^\omega \chi^{i+\omega} (p\cdot n)^j (p\cdot n^*)^l \times \frac{(-i|2i+\omega)(-j|-i-\omega)}{(1+l|i+\omega)}, \tag{11}$$

where

$$\chi \equiv \frac{2p\cdot np\cdot n^*}{n\cdot n^*},$$

and the superscript "AC" means that the exponents (i,j) were analytically continued to allow for negative values, $\omega = D/2$ and we use the Pochhammer symbol,

$$(a|b) \equiv (a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}. \tag{12}$$

Observe that l must take only positive values or zero since the Pochhammer symbol containing $\Gamma(1+l)$ was not analytically continued.

Consider now the second integral, the vectorial one, given in (5). For this case, let

$$G^\mu = \int \mathrm{d}^D q q^\mu \times exp\left[-\alpha(q-p)^2 - \beta(q\cdot n) - \gamma(q\cdot n^*)\right]. \tag{13}$$

Introducing the standard trick of substituting the q^μ factor for a derivative in p_μ [12], we obtain

$$\begin{aligned} G^\mu &= \left(\frac{\pi}{\alpha}\right)^\omega \frac{\mathrm{e}^{-\alpha p^2}}{2\alpha} \times \frac{\partial}{\partial p_\mu} exp\left[\alpha p^2 + \frac{\beta\gamma}{2\alpha}(n\cdot n^*) - \beta p^+ - \gamma p^-\right] \\ &= \left(p^\mu - \frac{\beta}{2\alpha}n^\mu - \frac{\gamma}{2\alpha}n^{*\mu}\right)\mathcal{G}_0, \end{aligned} \tag{14}$$

where $p^+ = p\cdot n$ and $p^- = p\cdot n^*$, as usual in the light-cone notation [6,7]. Also, let us define

$$\mathcal{G}_0 \equiv \left(\frac{\pi}{\alpha}\right)^\omega \exp\left[\frac{\beta\gamma}{2\alpha}(n\cdot n^*) - \beta p^+ - \gamma p^-\right]. \quad (15)$$

Now, Taylor expanding the exponential in (13),

$$G^\mu = \sum_{i,j,l=0}^{\infty} \frac{(-1)^{i+j+l}\alpha^i\beta^j\gamma^l}{i!j!l!} T_1^\mu(i,j,l), \quad (16)$$

and following the steps for the NDIM calculation [1] we finally get

$$T_1^{\mu,AC}(i,j,l) = V_1^\mu T_1^{\rm AC}(i,j,l), \quad (17)$$

where

$$V_1^\mu \equiv p^\mu - \left[\frac{(i+\omega)p^-}{(1+i+l+\omega)(n\cdot n^*)}\right] n^\mu - \left[\frac{(i+\omega)p^+}{(1+i+j+\omega)(n\cdot n^*)}\right] n^{*\mu}. \quad (18)$$

This result is in Euclidean space and it is valid for a positive dimension ($D = 2\omega > 0$), *negative* exponents (i,j) and for $l \geq 0$.

The second-rank tensor integral in (6) can be evaluated in a similar way. The only thing that need to be taken into account is that now a second derivative is called for and the calculation becomes lengthier. We only quote the final result:

$$T_1^{\mu\nu,\rm AC}(i,j,l) = V_1^{\mu\nu} T_1^{\rm AC}(i,j,l), \quad (19)$$

where

$$\begin{aligned} V_1^{\mu\nu} \equiv\ & p^\mu p^\nu - \left[\frac{(i+\omega)p^+p^-}{(1+i+j+\omega)(1+i+l+\omega)(n\cdot n^*)}\right] g^{\mu\nu} \\ & - \left[\frac{(i+\omega)p^-}{(1+i+l+\omega)(n\cdot n^*)}\right](p^\mu n^\nu + p^\nu n^\mu) \\ & - \left[\frac{(i+\omega)p^+}{(1+i+j+\omega)(n\cdot n^*)}\right](p^\mu n^{*\nu} + p^\nu n^{*\mu}) \\ & + \left[\frac{(i+\omega)(1+i+\omega)p^+p^-}{(1+i+j+\omega)(1+i+l+\omega)(n\cdot n^*)^2}\right] \\ & \times (n^\mu n^{*\nu} + n^\nu n^{*\mu}) \qquad (20) \\ & + \left[\frac{(i+\omega)(1+i+\omega)(p^-)^2}{(2+i+l+\omega)(1+i+l+\omega)(n\cdot n^*)^2}\right] n^\mu n^\nu \\ & + \left[\frac{(i+\omega)(1+i+\omega)(p^+)^2}{(2+i+j+\omega)(1+i+j+\omega)(n\cdot n^*)^2}\right] n^{*\mu} n^{*\nu}. \end{aligned}$$

It can be noted that for the particular case of $i = j = -1$ the pole piece for $\omega \to 2$ only arises in the scalar integral factor $T_1^{\rm AC}(i,j,l)$; see (11).

Now, let us consider the integrals $\{T_2\}$. These contain two scalar products with n_μ, but again they are harmless in the NDIM approach because their exponents, before analytic continuation, are positive. However, in the usual positive dimensional approach, such factors can become singular and prescriptions become a necessity. Yet prescriptions cannot handle products; one needs to use partial fractioning first. Thus, the recourse is to use the so-called "decomposition formulas" such as (see, for example, [6, 13])

$$\frac{1}{(k\cdot n)(p-k)\cdot n} = \frac{1}{p\cdot n}\left[\frac{1}{(p-k)\cdot n} + \frac{1}{k\cdot n}\right], \quad p\cdot n \neq 0, \quad (21)$$

NDIM does not require any of such partial fractionings; it can handle products at the same time. Not only that: NDIM can handle any power of these products simultaneously, i.e., factors of the form $(k\cdot n)^{-\alpha}[(p-k)\cdot n]^{-\beta}$, with $(\alpha,\beta = 2,3,\cdots)$ which, of course, become more strenuously difficult to handle by partial fractioning the higher the power we have.

To evaluate T_2 using NDIM, let us then consider the Gaussian-like integral,

$$G_2 = \int {\rm d}^D q \exp\left[-\alpha(q-p)^2 - \beta q\cdot n - \gamma(q-p)\cdot n - \delta q\cdot n^*\right], \quad (22)$$

which yields

$$G_2 = \left(\frac{\pi}{\alpha}\right)^{D/2} \exp\left(-\beta p^+ - \delta p^- + \frac{\beta\delta}{2\alpha} n\cdot n^* + \frac{\gamma\delta}{2\alpha} n\cdot n^*\right). \quad (23)$$

On the other hand, a direct Taylor expansion of (22) yields

$$G_2 = \sum_{i,j,l,m=0}^{\infty} (-1)^{i+j+l+m} \frac{\alpha^i\beta^j\gamma^l\delta^m}{i!j!l!m!} T_2(i,j,l,m). \quad (24)$$

Comparing both expressions and solving for $T_2(i,j,l,m)$ we get a unique solution for a system of 4×4 linear algebraic equations [2], which analytically continued to positive dimension and *negative* values for (i,j,l) finally gives

$$T_2^{\rm AC}(i,j,l,m) = \pi^\omega \chi^{i+\omega}(p^+)^{j+l}(p^-)^m \times \frac{(-i|2i+l+\omega)(-j|-i-l-\omega)}{(1+m|i+\omega)}. \quad (25)$$

Again, the superscript "AC" means that (i,j,l) is strictly *negative* and $m \geq 0$.

With the help of (23) it is easy to solve the two remaining integrals. The final results we quote here:

$$T_2^{\mu,AC}(i,j,l,m) = V_2^\mu T_2^{\rm AC}(i,j,l,m), \quad (26)$$

where

$$V_2^\mu \equiv p^\mu - \left[\frac{(i+\omega)p^-}{(1+i+m+\omega)(n\cdot n^*)}\right] n^\mu - \left[\frac{(i+l+\omega)p^+}{(1+i+j+l+\omega)(n\cdot n^*)}\right] n^{*\mu} \quad (27)$$

and

$$T_2^{\mu\nu,\mathrm{AC}}(i,j,l,m) = V_2^{\mu\nu} T_2^{\mathrm{AC}}(i,j,l,m), \tag{28}$$

where

$$\begin{aligned} V_2^{\mu\nu} \equiv{}& p^\mu p^\nu \\ & - \left[\frac{(i+l+\omega)p^+p^-}{(1+i+j+l+\omega)(1+i+m+\omega)(n\cdot n^*)}\right] g^{\mu\nu} \\ & - \left[\frac{(i+\omega)p^-}{(1+i+m+\omega)(n\cdot n^*)}\right](p^\mu n^\nu + p^\nu n^\mu) \\ & - \left[\frac{(i+l+\omega)p^+}{(1+i+j+l+\omega)(n\cdot n^*)}\right](p^\mu n^{*\nu} + p^\nu n^{*\mu}) \\ & + \left[\frac{(i+l+\omega)(1+i+\omega)p^+p^-}{(1+i+m+\omega)(1+i+j+l+\omega)(n\cdot n^*)^2}\right] \\ & \times (n^\mu n^{*\nu} + n^\nu n^{*\mu}) \\ & + \left[\frac{(i+\omega)(1+i+\omega)(p^-)^2}{(2+i+m+\omega)(1+i+m+\omega)(n\cdot n^*)^2}\right] n^\mu n^\nu \\ & + \left[\frac{(i+l+\omega)(1+i+l+\omega)(p^+)^2}{(2+i+j+l+\omega)(1+i+j+l+\omega)(n\cdot n^*)^2}\right] \\ & \times n^{*\mu} n^{*\nu}. \end{aligned} \tag{29}$$

Finally, before closing this section, let us analyze (7) with momentum shift $q = p - k$, so that

$$\begin{aligned} T_2(i,j,l,m) &= (-1)^{j+l+m} \tau_2(i,j,l,m), \quad \text{or} \\ \tau_2(i,j,l,m) &= (-1)^{-j-l-m} T_2(i,j,l,m), \end{aligned} \tag{30}$$

where

$$\tau_2(i,j,l,m) = \int \mathrm{d}^D k k^{2i} [(k-p)\cdot n]^j (k\cdot n)^l [(k-p)\cdot n^*]^m. \tag{31}$$

We can easily write down the following results:

$$\tau_2^\mu = p^\mu \tau_2 - (-1)^{-j-l-m} T_2^\mu, \tag{32}$$

and

$$\tau_2^{\mu\nu} = -p^\mu p^\nu \tau_2 + p^\mu \tau_2^\nu + p^\nu \tau_2^\mu + (-1)^{-j-l-m} T_2^{\mu\nu}. \tag{33}$$

The particular cases for T_1, T_2 and τ_2 such as $T_1(-1,-1,0)$, $T_2(-1,-1,-1,0)$, etc., can be worked out from the general expressions. All the above results are in agreement with the ones tabulated in [6,7,14].

3 Discussion and conclusion

NDIM is a technique wherein the principle of analytic continuation plays a key role. We solve a "Feynman-like" integral, i.e., a negative dimensional loop integral with propagators raised to positive powers in the numerator and then analytically continue the result to allow for negative values of those exponents and for a positive dimension.

In positive dimensions, Feynman integrals for covariant gauge choices can be worked out with a variety of methods. However, when we work in the light-cone gauge, things become more complicated by virtue of the presence of unwieldy gauge-dependent singularities. At this point NDIM turns out to have a surprising effect: propagators raised to positive powers in the "Feynman-like" integrals do not have poles of any kind to trouble us. *Therefore no prescription is needed* in the NDIM approach, and moreover, *no partial fractioning is necessary.* The beauty and the strength of NDIM to deal with light-cone integrals is revealed and demonstrated in a marvelous way.

So, we can summarize all this by enumerating the outstanding features of NDIM:

(1) No prescription at all is required to deal with gauge-dependent poles of the usual Feynman integrals.
(2) The overall structure of the Feynman integrals in the light-cone gauge is preserved, i.e., there is no need to introduce factors of the form $q\cdot n^*$ in the denominators as prescription input.
(3) There is no need to use a parametrization of any kind, so that there are no parametric integrals to solve.
(4) There is no need to perform integration with split components such as in [13], where the integration in space-time is performed by decomposing $\mathrm{d}^{2\omega}q \to \mathrm{d}^{2\omega-1}\mathbf{q}\mathrm{d}q_4$;.
(5) There is no need to resort to partial fractionings such as (21).
(6) The final result is obtained for arbitrary *negative* exponents of propagators, so that special cases of interest are all contained therein.
(7) The final result is already within the dimensional regularization context.

In this work we calculated integrals – scalar, vector, and second-rank tensor ones – pertaining to light-cone gauge with arbitrary exponents of the propagators and arbitrary dimension. Our results given in (11), (17), (19), (25), (26), (28), (30), (32) and (33) can be worked out for particular values for the exponents and compared to those existing in the literature. They can be checked to be in agreement.

But beyond doubt, the most outstanding conclusion that we can draw from this exercise is that *no prescription* was required to tackle the light-cone singularities. Of course, it is a matter of straightforward generalization that all other non-covariant gauge choices will follow suit.

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